CONTRIBUTION TO THE THEORY OF THE DETERMINATION OF THE ORBITAL ELEMENTS OF SOLAR SYSTEM BODIES: REPLACING GAUSS'S RATIO OF SECTOR TO TRIANGLE

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ABSTRACT
The aim of this paper is the determination of six orbital elements if two positions (1 and 2) of a solar system body are known. The task was solved using the ratio of the elliptical sector to the area of the corresponding triangle. This relationship was labelled as η and derived by K. F. Gauss (1809), see sections 1 and 2. This paper presents four easy methods that supply the ratio, see section 3. These methods simplify this task by removing long and very complicated derivations and by clarifying the theory and calculations. The methods are not used in the case of more than two observations.

KEYWORDS: celestial mechanics, orbital elements

1. INTRODUCTION

The presented paper is part of the sphere of theoretical astronomy in which the orbital elements are determined. The nature of the problem falls under celestial mechanics. The theory has become substantially easier as mutual gravitational attraction and other disturbances were not used. We shall deal with different applications of Kepler's laws, although in an ambiguous form. The derived equations can be used not only in the heliocentric system but also in any of the planetocentric systems. We shall deal with the determination of the elements of elliptical orbit. We shall not study parabolic and hyperbolic orbits. The fundamental plane is not only ecliptic but more often the plane of the equator of the central body. The task that shall be solved in the following text is:

Determine six orbital elements of a solar system body if its two centric positions 1 and 2 in the given coordinate system are known!

The first solution of the task was done by Gauss using the ratio

η = (the area of the elliptical sector)/(the area of the corresponding triangle),

see Gauss (1809). The solution of the task has been modified by many authors. For the modifications as well as the history of the determination of the orbital elements in detail see M. F. Subbotin (1968), particularly G. Stracke (1929) and recently D. L. Boulet (1991). A bibliography of the works on the classical method of the determination of the orbits of planets and comets published before 1900 is given in Radau (1899). The bibliography between 1900 and 1928 is shown in Stracke (1929) and in Subbotin (1968).

2. THE BASIC IDEAS OF THE DETERMINATION OF THE ORBITAL ELEMENTS USING THE RATIO *η*

Gauss's original method consists of the derivation of the equation

$$\eta = \frac{c(t_2 - t_1)}{r_1 r_2 \sin(v_2 - v_1)}\,, \tag{1}$$

which is the sought Gauss's ratio of sector to triangle (Gauss, 1809). Here c denotes the double of the sectorial velocity and r_1, r_2, v_1, v_2 (see Fig. 1), are the geocentric radii and the true anomalies in the times t_1 and t_2, respectively. The sought ratio is also valid for other bodies which are gravitationally attracted to each other. Gauss was led to the equation by the task of determining the orbit using two given geocentric positions in both polar and rectangular coordinates. Using η we get the parameter of the elliptical orbit (the semi latus rectum of a conic) p immediately from the equation $p = \left[\frac{\eta}{k(t_2 - t_1)} r_1 r_2 \sin(v_2 - v_1)\right]^2$ and then easily and successively we get the remaining orbital elements; k is Gauss's gravitational constant. The derivation of the ratio η is very difficult and time consuming. It leads to the equations

$$x = \frac{m}{\eta^2} - l$$
$$X = \frac{4}{3} + \frac{4.6}{3.5}x + \frac{4.6.8}{3.5.7}x^2 + \cdots \tag{2}$$
$$\eta = 1 + X(l + x)$$

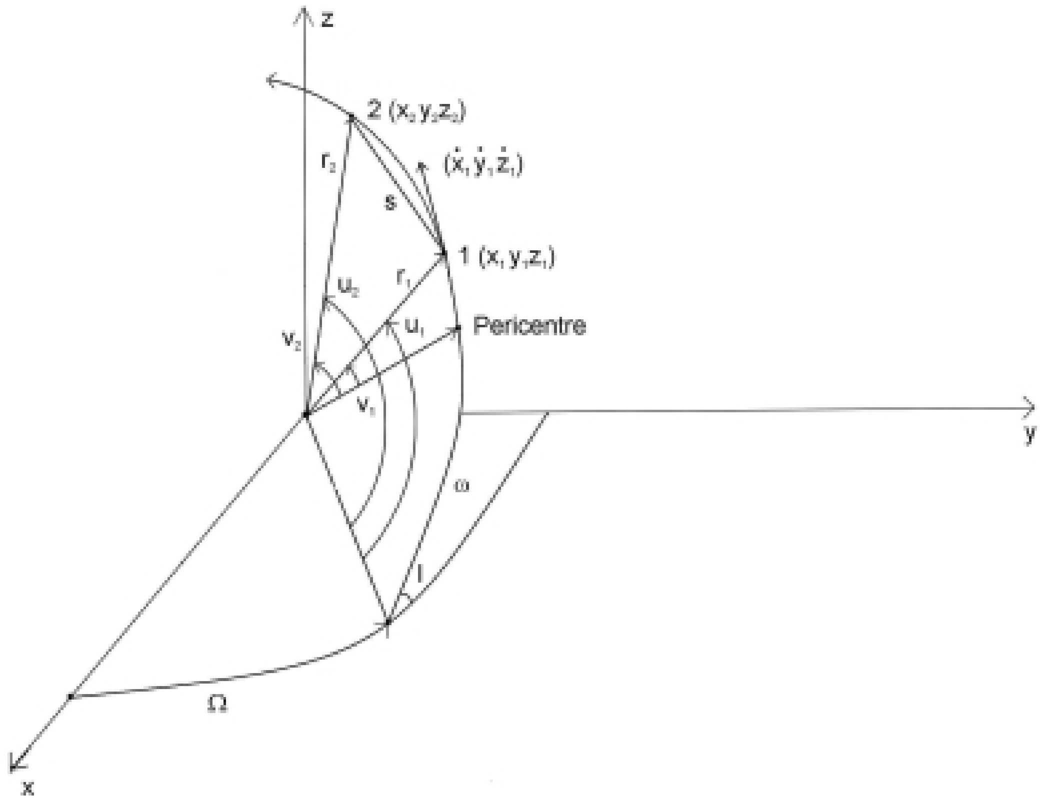

Fig. 1 The orbit of a solar system body and its two centric positions at epochs t_1, t_2.

Here

$$m = k^2 \frac{(t_2 - t_1)^2}{\left(2\sqrt{r_1 r_2}\cos\frac{v_2 - v_1}{2}\right)^2},$$

$$l = \frac{(r_1 + r_2)}{4\sqrt{r_1 r_2}\cos\frac{1}{2}(v_2 - v_1)} - \frac{1}{2}$$

are the quantities which we get from the given values. The task involves successive approximations, that is to say the solution is not direct. The initial x is derived from the first Eq. (2) using $\eta = 1$.

The direct solution of the ratio η was done later by P.A. Hansen using some simplifications. First of all, he supposes x to be a very small value and uses absolute and linear terms only. In the following arrangement he omits quantities of the second and higher orders and gets what is called Hansen's recurrent equation, or Hansen's chain fraction; for both see Stracke (1929), page 27. The solution of the ratio η has been simplified but once again by using greatly simplified successive approximations.

The method proposed by Lambert led to Lambert's theorem (Lambert, 1902 and Stracke, 1929), which also requires the use of successive approximations or iterations and is therefore an indirect method. A special case of Lambert's theorem is Euler's theorem, see Subbotin (1968) page 160. From Lambert's theorem we arrive at Gauss's ratio η.

The methods mentioned in this section are the oldest ones. They were proposed when the branch of orbital element determination was being formulated. The methods are complicated from the theoretical point of view, e.g. their derivation. Nevertheless they need iterations either of approximation or interpolation.

3. OTHER METHODS WITHOUT USING THE RATIO *η*

The task is the same as is described in the Introduction. We shall show the recent methods and untraditional methods that use new theoretical relations of the problem of two bodies but also use modern computers. This is the author's primary intention. In the following considerations we established that the origin of the coordinate system is identical to the focus of the elliptical orbit of the studied body; the fundamental plane *x, y* is identical to the equatorial plane of the central body (for example the Earth in the case of artificial satellites) and the fundamental direction is the direction to the vernal point. The *z* axis complements the system to the right hand and orthogonal system. The two centric positions used, 1 and 2, are defined by the coordinates x_i, y_i, z_i at the times t_i for i=1, 2, see Fig. 1. The Gaussian gravitational constant $k = (GM)^{1/2}$.

The cyclic methods are shown in sections 3.2.1 and 3.2.2, where no modifications of the equations of the problem of two bodies are applied and the one determined element is specified gradually.

The solution using derivation by a selected element will be shown in section 3.2.3. This is an older method which complements the two methods described above and is here presented in a shorter form.

In section 3.2.4 the improvements of the velocity components at point 1 are specified. The orbital task with the two known positions 1 and 2 will be so modified to the osculating method with the known position and velocity at one position 1 only. The solution is then easier. None of the four cases use the ratio η, Lambert's theorem, or any other modifications.

3.1. DESCRIPTION OF THE CALCULATION COMMON TO ALL OF THE FOLLOWING CASES

The lengths of the radius vectors r_1, r_2 and the difference of the true anomalies $v_2 - v_1$ are obtained directly from the given data, see Fig. 1. The right ascension of the ascending node Ω, the inclination of the orbital plane I, and the arguments of declination u_1 and u_2 from the equations

$$\mathrm{tg}\,\Omega = \frac{\mathrm{tg}\,\delta_1 \sin\alpha_2 - \mathrm{tg}\,\delta_2 \sin\alpha_1}{\mathrm{tg}\,\delta_1 \cos\alpha_2 - \mathrm{tg}\,\delta_2 \cos\alpha_1}$$

$$\mathrm{tg}\, I = \frac{\mathrm{tg}\,\delta_i}{\sin(\alpha_i - \Omega)} \qquad (3)$$

$$\sin u_i = \frac{\sin\delta_i}{\sin I}, \qquad \cos u_i = \frac{\cos\delta_i}{\cos(\alpha_i - \Omega)}$$

where the relation tg $\alpha_i = y_i / x_i$, sin $\delta_i = z_i / r_i$ and $r_i = (x_i^2 + y_i^2 + z_i^2)^{1/2}$ for i = 1 and 2 apply for right ascension, declination, respectively. We shall use the relation

$$L = k(t_2 - t_1) \quad [\mathrm{m}^{3/2}]. \qquad (4)$$

We can also derive the length of the chord 1-2, see Fig. 1. The relations shown above give the final values and the following calculations have no influence on them. Thus we know two orbital elements, *Ω* and *I*, which define the orientation of the orbital plane in the reference system used. We also have to determine the four remaining elements. Therefore we shall choose the approximate value of the argument *ω* of the pericentre, which orientates the orbital ellipse in the orbital plane with respect to the ascending node, see Fig. 1. We shall then gradually elaborate upon it. After obtaining the required precision we calculate the remaining elements.

The true anomalies are determined using

$$v_i = u_i - \omega \qquad (5)$$

where i = 1 and 2 for the times t_1 and t_2. The eccentricity is derived from the true anomalies

$$e = (r_2 - r_1)/(r_1 \cos v_1 - r_2 \cos v_2) \qquad (6)$$

and we can use two different ways to get the semi major axis

$$a_i = r_i(1 + e\cos v_i)/(1 - e^2) \qquad (7)$$

and the eccentric and mean anomalies E_i and M_i with the help of

$$\mathrm{tg}\frac{E_i}{2} = \sqrt{\frac{1-e}{1+e}}\,\mathrm{tg}\frac{v_i}{2}, \quad M_i = E_i - e\sin E_i. \qquad (8)$$

The value *L* in the Eq. (4) is derived exactly. We can also express the value *L* in another way

$$P_i = (a_i)^{3/2}(M_2 - M_1). \qquad (9)$$

This depends on the *ω* chosen. Comparing Eqs. (4) and (9) we get

$$\Delta_i = L - P_i, \quad [\mathrm{m}^{3/2}]. \qquad (10)$$

If *ω* is exact then $\Delta_i = 0$ and this is the key to the next successive iterations of the Eqs. (5) - (10) and in this way also the improvement of the elements *a, e* and M_0. This way is valid for the first three methods in section 3.2. The calculation is repeated until Δ_i, see Eq. (10), is given more precisely.

The fourth method, 3.2.4, is similar but the successively improved values are the components of the velocity at point 1, see Fig. 1.

The input values of the illustration example are as follows:

x_1 = 10 000 000.23 m,
y_1 = 39 999 999.987 m,
z_1 = 5 000 000.006 m,
x_2 = 4 316 743.858 616 m,
y_2 = 42 181 800.563 998 m,
z_2 = -5 183 743.556 899 m,
t_1 = 0 h, t_2 = 1 h, $k^2 = GM$ = 3.986 004 415 ×10^{14} $\mathrm{m}^3/\mathrm{s}^2$

and to check we also show the correct values of the results for time $t_0 = t_1 = 0$ h:

Ω = 173°. 290 163 212 887 6,
a = 25 015 181.040 748 56,
I = 6°. 970 729 214 976 ,
e = 0.707 977 170 849 52,
ω = 91°.552 886 987 917 7,
M_0 = 144°.224 991 298 787 8.

3.2. NEW PROPOSED METHODS

3.2.1. METHOD OF THE SUCCESSIVELY IMPROVING ARGUMENT Ω OF THE PERICENTER

1. We determine the interval *Δω* (in the range 0-180°) and calculate Δ_j (the index *i* in the Eq. (10) has been changed to the index *j*) with the chosen step k_ω of the element *ω*. The index *j* then corresponds to the value that has been calculated in the steps k_ω.

Table 1 Specifying the argument ω of the pericenter using the reduction of its interval $\Delta\omega$.

ω [°]	Δ_j, Eq. (10) [$m^{3/2}$]
$\Delta\omega$ from 91° to 92° with the step 0.01°	
91.55000000000028	-15773185.27116394
91.56000000000029	38794322.81787109
$\Delta\omega$ from 91.55° to 91.56° with the step 0.0001°	
91.55280000000009	-475029.3886260986
91.55290000000009	71055.99824523926
$\Delta\omega$ from 91.5528° to 91.5529° with the step 0.000001°	
91.55288699999979	-5394.806854248047
91.55288699999979	65.97738647460937
$\Delta\omega$ from 91.552886° to 91.552887° with the step 0.00000001°	
91.55288697999939	-43.24050903320312
91.55288698999938	11.36734008789062
$\Delta\omega$ from 91.55288698° to 91.55288699° with the step 10^{-10}°	
91.55288698790014	-.0962371826171875
91.55288698800014	.4498291015625

2. As the correct value is valid for $\Delta_j = 0$, we take the two values closest to zero out of the table, one positive and the second negative.
3. These two values define the new reduced interval $\Delta\omega$ which is used to form a new table with the new reduced step $k_\omega/100$, etc.

The tabulation is reiterated until the required accuracy is achieved. The accuracy is checked using the Δ_j closest to zero. The first and second equations of Eq. (3) have been used to determine Ω and I. The other values are then improved successively using Eqs. (4) - (10).

The resulting value $\omega = 91.^{o}552\ 886\ 988$ with the accuracy (absolut) 10^{-11}. The number of significant digits is too high to be of use. It is used only to verify the method. The accuracy of the other elements is also suitable, see the end of section 3.1.

3.2.2. METHOD USING CHORDS.

The Eqs. (3) and (4) are again used to calculate the right ascension of the ascending node Ω, the orbit inclination I, the declination arguments u_1, u_2 and the relation of L. Now, see Fig. 2 and the first two lines in Tab. 2; the boundary ω_N and ω_V has been chosen as the points A_N and A_V on both sides of the axis ω. The function $\Delta = \Delta(\omega)$ needs to be known at least approximately. The angular coefficient (slope) of the chord $\overline{A_N, A_V}$ is

$$ch = \frac{\Delta_V - \Delta_N}{\omega_V - \omega_N} \tag{11}$$

and the corresponding part on the axis ω is

$$\omega_P = \omega_V - \Delta_V / ch\,. \tag{12}$$

And again, the $\Delta_P = \Delta_P(\omega_P)$ is calculated using the procedure described in section 3.2.1. The point A_P takes on the role of point A_N, see Fig. 2, so that in Eq. (11) $\Delta_N \to \Delta_P$ and $\omega_N \to \omega_P$. We get the new value ch of the angular coefficient and the new value ω_P from Eqs (11) and (12). The method continues successively until the point A_P is close to the axis ω within the required range of accuracy or is even identical to it. The procedure is similar to the Newton's iteration, but it is not necessary to derive and calculate complicated derivatives. The next lines in the Tab. 2 show ω_P and Δ_P obtained from each iteration. The resulting value ω_P = 91.552 886 988° was taken from the last line of Tab. 2 taking into account the desired accuracy 10^{-6}.

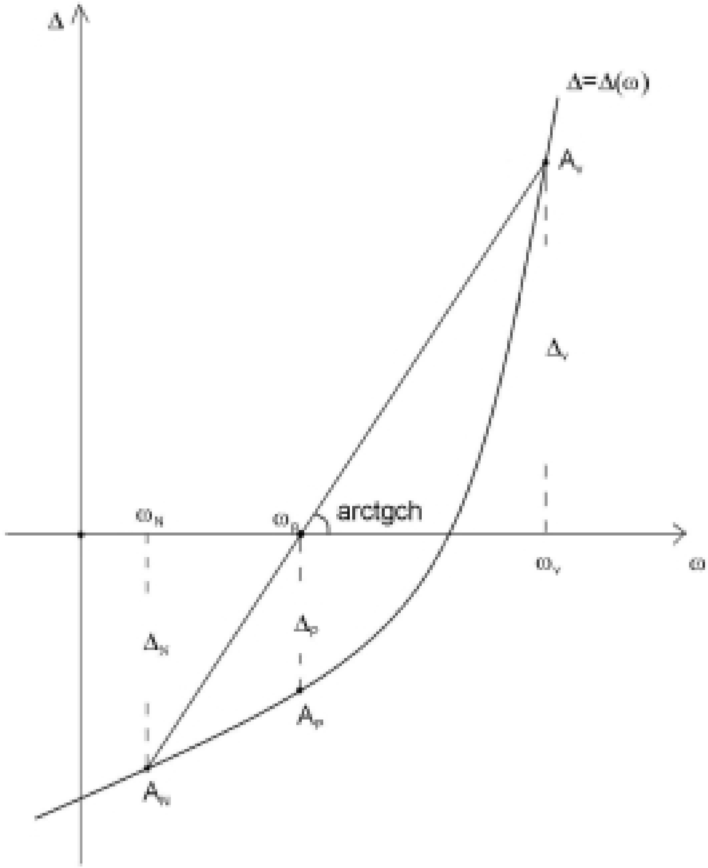

Fig. 2 The dependence of the difference $\Delta_i = L - P_i$ on the argument of pericenter ω.

Table 2 Specifying the argument of perifocus ω using the chord method.

limits of ω [°]	Δ_i, Eq. (10) [$m^{3/2}$]
91.55000000000028	-15773185.27116394
91.56000000000029	38794322.81787109

improving of ω_P [°]	$\Delta_{Pj} = \Delta_i$, Eq. (10) [$m^{3/2}$]
91.55000000000028	-15773185.27116394
91.5528905819278	19626.09710693359
91.5528869834437	-24.43177795410156
91.55288698792332	0.0304412841796875

the resulting elements:

Ω [°]	ω [°]	I [°]:
173.2901632128876	91.55288698792332	6.970729214976468
a [m]	e	M_o [°]:
25015181.0406991	0.7079771708528388	144.2249912987904

in the time [h] 0.

The resulting ω_P corresponds to the correct value and greatly exceeds the desired accuracy. Even the remaining Kepler's elements correspond to the correct values.

3.2.3. METHOD USING DERIVATIONS

First we again derive the right ascension Ω of the ascending node, the inclination I of the orbit, and both arguments of declination u_1 and u_2 from Eqs (3). Then we prepare the approximation in a similar manner to that used in the previous section. We shall use $L = B(\omega)$, see also Eq. (4), where the left side of the equation is known and is expressible using the input values. The right side contains the true anomaly ω, and numerical eccentricity e, see Eqs. (5) and (6). We introduce $\omega = \omega_0 + \Delta\omega$ and the Eq. (4) is expanded to the Taylor series using the linear term only

$$L = B(\omega_0) + \left.\frac{dB(\omega)}{d\omega}\right|_{\omega=\omega_0} .\Delta\omega \tag{13}$$

From the Eq. (13) we get the improving term $\Delta\omega$ and then the improved ω. The derivation of $B(\omega)$ exists as well as the convergency. The calculation is repeated starting from Eq.(5) and ending with Eq. (13) till the desired accuracy is achieved. For more about the method, including the required derivations and example, see Kabeláč (1967).

3.2.4. METHOD USING THE IMPROVEMENT OF THE VELOCITY COMPONENTS.

We start from the highly important equations mentioned, for example, in Subbotin (1968). They are

$$x_2 = x_1.F + \dot{x}_1.G, \quad y_2 = y_1.F + \dot{y}_1.G, \quad z_2 = z_1.F + \dot{z}_1.G \tag{14}$$

Then

$$F = 1 - \frac{2a}{r_1}\sin^2\frac{E_2 - E_1}{2},$$

$$G = \frac{1}{n}\left[\sin(E_2 - E_1) - (E_2 - E_1)\right] + (t_2 - t_1) . \tag{15}$$

Here the relations with index 1 correspond with the time t_1 and the relations with index 2 with the time t_2 for which the coordinates x_2, y_2, z_2 are calculated, see Boulet (1991), Kabeláč (1989), Subbotin (1968). Note that the relations (14) and (15) are final (closed) and are valid for any time interval $t_2 - t_1$. This means that the relations are not a development retaining only the first two terms.

We need to modify the Eqs. (14) to the form

$$\dot{x}_1 = (x_2 - x_1.F)/G, \quad \dot{y}_1 = (y_2 - y_1.F)/G, \quad \dot{z}_1 = (z_2 - z_1.F)/G \tag{16}$$

for the calculation of the velocity components at point 1, if the coordinates of points 1 and 2 are known. So the task then becomes the easier task of determining the osculating elements. Let us repeat the task before presenting the method. We have the centric coordinates x_1, y_1, z_1 and x_2, y_2, z_2 valid for the moments t_1 and t_2 and the geocentric gravity constant $k = (GM)^{1/2}$.

First approximation

$\mathbf{r}_i = (x_i^2 + y_i^2 + z_i^2)^{1/2}$: i=1,2, [m], radius vectors

$\cos(v_2 - v_1) = (x_1x_2 + y_1y_2 + z_1z_2)/r_1r_2$, [0], difference of true anomalies

$n^{1.} = (v_2 - v_1)/(t_2 - t_1)$, [rad/s], mean angular velocity

$a^{1.} = [GM/(n^{1.})^2]^{1/3}$, [m], semi major axis

$F^{1.} = \dfrac{2a^{1.}}{r_1} - \sin^2\dfrac{v_2 - v_1}{2}$, [0], auxiliary value

$G^{1.} = \dfrac{1}{n^{1.}}[\sin(v_2 - v_1) - (v_2 - v_1)] + (t_2 - t_1)$, [s], auxiliary value

$$\dot{x}_1^{1.} = (x_2 - x_1.F^{1.})/G^{1.},$$
$$\dot{y}_1^{1.} = (y_2 - y_1.F^{1.})/G^{1.}, \qquad \text{[m/s]}$$
$$\dot{z}_1^{1.} = (z_2 - z_1.F^{1.})/G^{1.},$$

Here the upper index 1. means the first approximation. The values of r_i and $v_2 - v_1$ are final.

Second and further approximations:

$V_1^{2.} = [(\dot{x}_1^{1.})^2 + (\dot{y}_1^{1.})^2 + (\dot{z}_1^{1.})^2]^{1/2}$, [m/s], velocity

$(r_1\dot{r}_1)^{2.} = x_1\dot{x}_1^{1.} + y_1\dot{y}_1^{1.} + z_1\dot{z}_1^{1.}$, [m^2/s], auxiliary value

$W^2 = \dfrac{2GM}{r_1} - (V_1^{2.})^2$, [m^2/s^2], auxiliary value

$n^{2.} = (W^{2.})^{3/2}/GM$, [s^{-1}], mean angular velocity

$a^{2.} = GM/W^{2.}$, [m], semi major axis

$(e\sin E_1)^{2.} = \dfrac{(r_1\dot{r}_1)^{2.}}{GM}(W^{2.})^{1/2}$, [0], numerical eccentricity e and accentric anomaly E_1

$(e\cos E_1) = 1 - r_1/a^{2.}$, [0], numerical eccentricity e and accentric anomaly E_1

$$(E_2 - E_1)^{2.} = n^{2.}(t_2 - t_1) + \sin(E_2 - E_1)^{1.}(e\cos E_1)^{2.} + [\cos(E_2 - E_1)^{1.} - 1](e\sin E_1)^{2.},$$

where

$$(E_2 - E_1)^{1.} = n^{2.}(t_2 - t_1)$$

$F^{2.} = 1 - \dfrac{2a^{2.}}{r_1}\sin^2\dfrac{(E_2 - E_1)^{2.}}{2}$, [0], auxiliary value

Table 3 Improving the velocity components [m/s] after Eq. (16)

Approx.	$\dot{x}_1$	$\dot{y}_1$	$\dot{z}_1$
1	-1541.1187587180	779.6068826596	-72.6505679070
2	-1498.6425154536	1007.7789911214	-100.9639845775
3	-1500.0469219373	999.7315323979	-99.9667302104
4	-1499.9983727776	1000.0092781402	-100.0011500386
5	-1500.0000495137	999.9996851269	-99.9999612193
6	-1499.9999916026	1000.0000164498	-100.0000022787
7	-1499.9999936027	1000.0000050067	-100.0000008606
8	-1499.9999935336	1000.0000054018	-100.0000009095
Final	**-1499.99999341**	**1000.00000534**	**-100.0000086**

And finally the velocity components

$$\dot{x}_1^{2.} = (x_2 - x_1.F^{2.}) / G^{2.},$$
$$\dot{y}_1^{2.} = (y_2 - y_1.F^{2.}) / G^{2.}, \qquad \text{[m/s]}$$
$$\dot{z}_1^{2.} = (z_2 - z_1.F^{2.}) / G^{2.},$$

The last relations for the second approximation are repeated to reach the required accuracy. The upper index $^{2.}$ changes for $^{3.}$ and so on. The numerical values of the velocity components after each approximation are shown in Tab. 3. The search accuracy of the velocity components is 10^{-6}, as proved by the results.

4. CONCLUSION

We intended to show several simple ways of solving the task of determining the orbital elements if two centric positions of the body are known without using the ratio η triangle to sector. These simplified methods fully replace the ratio η that was introduced by K. F. Gauss 200 years ago and meant the solution to one of the basic orbital tasks. In subsequent years the ratio was used even using later modifications. The 4 methods presented here fully replace the ratio, being even simpler and consequently more understandable than Gauss's original derivation. The last method, presented in section 3.2.4, is based on relations which are not familiar.

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